Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 3

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

7 pages

2223 – 7208

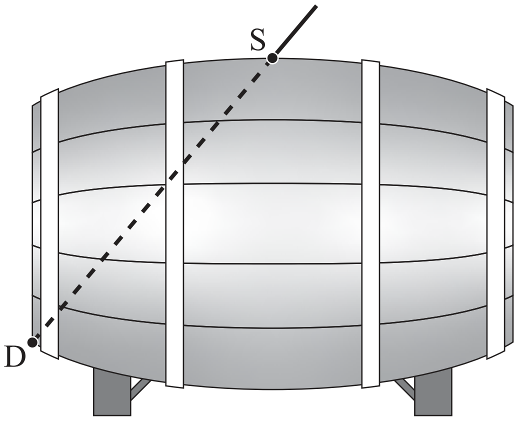
Blank page

Answer **both** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 26]

In this question you will use a historic method of calculating the cost of a barrel of wine to determine which shape of barrel gives the best value for money.

In Austria in the 17th century, one method for measuring the volume of a barrel of wine, and hence determining its cost, was by inserting a straight stick into a hole in the side, as shown in the following diagram, and measuring the length SD. The longer the length, the greater the cost to the customer.



Let SD be d metres and the cost be C gulden (the local currency at the time). When the length of SD was 0.5 metres, the cost was 0.80 gulden.

(a) Given that C was directly proportional to d, find an equation for C in terms of d. [3]

A particular barrel of wine cost 0.96 gulden.

(b) Show that $d = 0.6$. [1]

(This question continues on the following page)

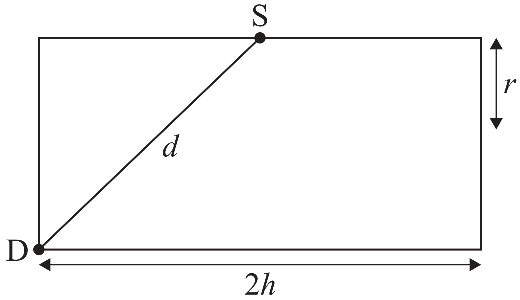
Turn over

(Question 1 continued)

This method of determining the cost was noticed by a mathematician, Kepler, who decided to try to calculate the dimensions of a barrel which would give the maximum volume of wine for a given length SD.

Initially he modelled the barrel as a cylinder, with S at the midpoint of one side. He took the length of the cylinder as $2h$ metres and its radius as r metres, as shown in the following diagram of the cross-section.

diagram not to scale



(c) Find an expression for r^2 in terms of d and h. [3]

Let the volume of this barrel be $V\ \text{m}^3$.

(d) Show that $V = \frac{\pi}{2}\left(d^2h - h^3\right)$. [2]

The remainder of this question considers the shape of barrel that gives the best value when $d = 0.6$.

(e) (i) Use the formula from part (d) to find the volume of this barrel when $h = 0.4$. [2]

(ii) Use differentiation to show that $h = \sqrt{0.12}$ when $\frac{dV}{dh} = 0$. [3]

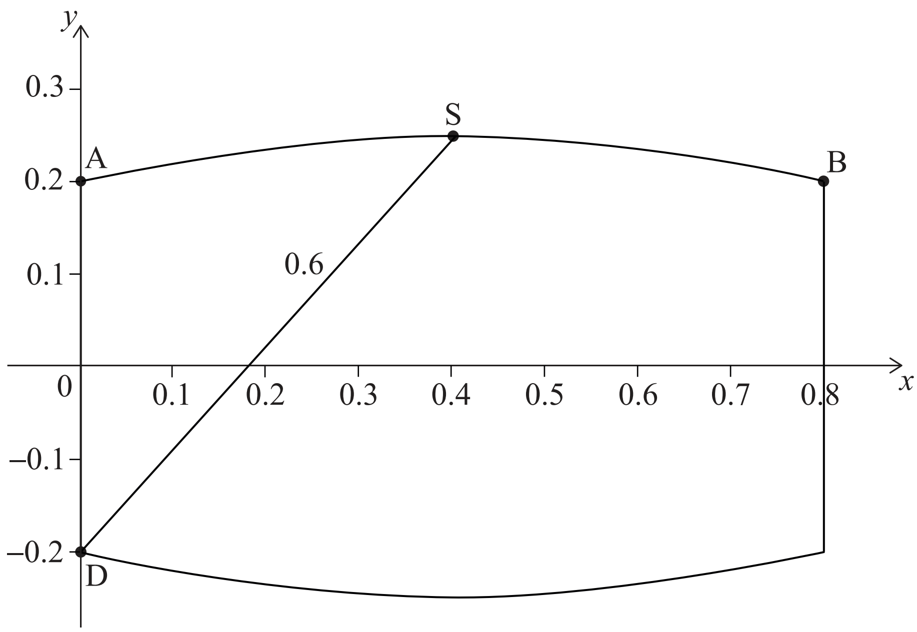
(iii) Given that this value of h maximizes the volume, find the largest possible volume of this barrel. [2]

(This question continues on the following page)

(Question 1 continued)

Kepler then considered a non-cylindrical barrel whose base and lid are circles with radius $0.2\,\text{m}$ and whose length is $0.8\,\text{m}$.

He modelled the curved surface of this barrel by rotating a quadratic curve, ASB, with equation $y = ax^2 + bx + c$, $0 \le x \le 0.8$, about the x-axis. The origin of the coordinate system is at the centre of one of the circular faces as shown in the following diagram. S is at the vertex of the quadratic curve and $\text{SD} = 0.6$.



Kepler wished to find out if his barrel would give him more wine than any cylindrical barrel with $d = 0.6$.

The coordinates of A and B are $(0,\ 0.2)$ and $(0.8,\ 0.2)$ respectively.

(f) Find the equation of the quadratic curve, ASB. [6]

(g) Show that the volume of this barrel is greater than the maximum volume of any cylindrical barrel with $d = 0.6$. [3]

(h) State one assumption, not already given, that has been made in using these models to find the shape of the barrel that gives the best value. [1]

2. [Maximum mark: 29]

In this question you will use vector methods to determine whether aircraft are obeying air traffic regulations.

The base of an air traffic control tower at an airport is taken as the origin of a coordinate system. An aircraft's position is given by the coordinates (x, y, z), where x and y are respectively the aircraft's displacement east and north of the tower, and z is the vertical displacement of the aircraft above the base of the tower. All displacements are measured in kilometres.

At $12:00$ two aircraft, A and B, are at the points $\text{P}(100, -82, 10.7)$ and $\text{Q}(215, -197, 10.7)$ respectively.

(a) Find the distance between the two aircraft at $12:00$. [2]

The two aircraft are flying along the same straight line (flight path), with B behind A.

They both have the same constant velocity of $\begin{pmatrix} -640 \\ 640 \\ 0 \end{pmatrix}$ kilometres per hour.

(b) Find the speed of both aircraft. [2]

Air traffic regulations state that if two aircraft are on the same flight path then they must always maintain at least a 10 minute gap between them. If at any time two aircraft are too close they are said to be "in conflict".

(c) Find the length of time it takes B to reach point P from point Q, and hence state whether the two aircraft are in conflict. [3]

(d) Write down, $\boldsymbol{r}_A$, the position vector of A, t hours after $12:00$. [1]

If two aircraft are not on the same flight path, air traffic regulations state:

When the vertical distance between the two aircraft is less than $300\,\text{m}$ the aircraft must be more than $10\,\text{km}$ apart.

When the vertical distance between the two aircraft is at least $300\,\text{m}$ there are no restrictions.

The air traffic controller notices an aircraft, C, flying on a different flight path but close to A. The position of C, t hours after $12:00$, is given by

$$\boldsymbol{r}_C = \begin{pmatrix} -400 \\ -41 \\ 9.1 \end{pmatrix} + t\begin{pmatrix} -140 \\ 604 \\ 2 \end{pmatrix}.$$

(This question continues on the following page)

(Question 2 continued)

(e) (i) Find the two values of t at which the distance between A and C is $10\,\text{km}$. [5]

It is given that the distance between A and C is less than $10\,\text{km}$, only between these two values of t.

(ii) Determine whether the two aircraft, A and C, will break the air traffic regulations if they continue with their current velocities. Justify your answer. [5]

A new coordinate system, (x, y), is defined with an origin R at a point $2\,\text{km}$ directly above the air traffic control tower. When an aircraft is flying in the horizontal plane containing R, the values of x and y represent its displacement, in km, east and north of point R respectively.

A fourth aircraft, D, is flying at a constant height of $2\,\text{km}$ near the airport while waiting for permission to land. Its position at time t is given by

$$\vec{\text{RD}} = \begin{pmatrix} 6.4\cos(38t) \\ 6.4\sin(38t) \end{pmatrix}.$$

(f) Describe the path followed by D. [3]

Around the same time a small aircraft, E, is flying at the same height as D and along the line with vector equation

$$\vec{\text{RE}} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(g) Let $\boldsymbol{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(i) Find $\vec{\text{RE}} \bullet \boldsymbol{b}$ in terms of λ. [2]

(ii) Hence find the value of λ for which the distance from R to the line is minimum. [2]

(iii) Find this minimum distance. [2]

(iv) Show that D and E will not break the air traffic regulations for any value of t. [2]

References: